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Ridder, A.

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Ad Ridder

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Ad Ridder

Free University of Amsterdam [1]

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[1]Full address: Dept. of Economy and Econometrics, De Boelelaan 1105, 1081 HV Amsterdam, the Netherlands, phone: (+31) 20 548 7070, fax: (+31) 20 646 1440, email: aridder@sara.nl

Abstract

In this paper we present the common approaches of defining stochastic orderings of probability measures on (multi-dimensional) lattices. Also we consider stochastic orderings of matrices of transition probabilities. Relationships and (counter)examples are given to illustrate the concepts.

1 Introduction

The purpose of this paper is to present an overview of two commonly used ways of introducing stochastic ordering of probability measures. Particularly how they relate to each other and what the consequences are for ordering relations of stochastic matrices, are points of study.

In this paper S is a d-dimensional lattice, usually $S = \prod_{k=1}^{d}\{0,1,\ldots,s_k\}$ or $S = \{0,1,\ldots\}^d$. The space of all probability measures on S is denoted by $\mathcal{M}$ and the space of all stochastic matrices on $S \times S$ by $\mathcal{S}$. The space of all nonnegative functions (or vectors) on S is $\mathcal{V} = \mathbb{R}^S_{\geq 0}$. Throughout the paper we shall frequently refer to different orderings on different spaces. Discrimination is made by use of a subscript: $\leq_S$ is some partial ordering on the lattice, for instance the vector ordering $\leq_v$; $\leq_{\mathcal{V}}$ is the vector ordering on the vectorspace $\mathcal{V}$; $\leq_{\mathcal{M}}$ is some stochastic ordering on the space of probability measures. No subscript is written when the usual linear ordering of real numbers is applicable.

2 Stochastic Ordering of Probabilities

Let $\mathcal{F} \subset \mathcal{V}$ be a class of nonnegative functions on S, and $\mathcal{A} \subset 2^S$ a collection of subsets of S. We avoid complications by assuming that the functions $f \in \mathcal{F}$ are nonnegative, hence the integrals with respect to probabilities $p \in \mathcal{M}$ always exist, and we write

$$pf = \int_S f dp$$

Actually the integral is a summation over all states of the (denumerable) S. Notice also that we do not have to worry about measurability of sets $A \in \mathcal{A}$. Two approaches for introducing stochastic orderings are classical and are called respectively integral and set stochastic orderings. We recall them here for notational purposes. One may find more details in [5] .

Definition 1 *For $p, q \in \mathcal{M}$.*

(i)

$$p \leq_{\mathcal{F}} q \Leftrightarrow pf \leq qf \quad \forall f \in \mathcal{F}$$

(ii)

$$p \leq_{\mathcal{A}} q \Leftrightarrow p(A) \leq q(A) \quad \forall A \in \mathcal{A}$$

The set stochastic ordering (ii) may be viewed as an integral stochastic ordering by

$$\mathcal{F} = \{1\{A\} : a \in \mathcal{A}\}$$

but it is more convenient to use definition 1. In case of (i) we say that "p is $\mathcal{F}$-stochastically smaller than q", and in case of (ii) "p is $\mathcal{A}$-stochastically smaller than q". It is easy to state relationships between these two orderings. We say that the functions $f \in \mathcal{F}$ are generated by the sets $A \in \mathcal{A}$ if they are linear combinations of indicator functions:

$$\mathcal{F} \subset \Omega(\mathcal{A}) \text{ if } \left(f \in \mathcal{F} \Rightarrow f = \sum_{k=0}^{\infty} a_k 1\{A_k\} \;\; a_k \geq 0, A_k \in \mathcal{A} \right)$$

When $\mathcal{F}$ is determined by taking arbitrary linear combinations of indicator functions we write $\mathcal{F} = \Omega(\mathcal{A})$. Hence,

$$\mathcal{F} = \Omega(\mathcal{A}) \text{ if } \left(f \in \mathcal{F} \Leftrightarrow f = \sum_{k=0}^{\infty} a_k 1\{A_k\} \;\; a_k \geq 0, A_k \in \mathcal{A} \right)$$

And we say that $\mathcal{A}$ is induced by $\mathcal{F}$ if the latter contains all the indicator functions:

$$\mathcal{A} \subset \Omega(\mathcal{F}) \text{ if } (A \in \mathcal{A} \Rightarrow 1\{A\} \in \mathcal{F})$$

Then

Corollary 1 *For $p, q \in \mathcal{M}$.*

(i)

$$\mathcal{F} \subset \Omega(\mathcal{A}) \Rightarrow (p \leq_{\mathcal{A}} q \Rightarrow p \leq_{\mathcal{F}} q)$$

(ii)

$$\mathcal{A} \subset \Omega(\mathcal{F}) \Rightarrow (p \leq_{\mathcal{F}} q \Rightarrow p \leq_{\mathcal{A}} q)$$

3 Monotonicity of Stochastic Matrices

We need ordering relations of stochastic matrices for later use when we apply these concepts to Markov chains. Notice that $Pf \in \mathcal{V}$ for any $P \in \mathcal{S}$ and $f \in \mathcal{F}$ by

$$Pf(x) = \int_S f(s)P(x, ds) \quad x \in S$$

and that $pP \in \mathcal{M}$ for any $p \in \mathcal{M}$ by

$$pP(x) = \int_S P(s, x)p(ds) \quad x \in S$$

First we shall define monotonicity of matrices, in the next section we shall focus on comparability. Again we refer to [5] for more details. Other 'standard' studies are [2, 3, 6].

Definition 2 *Let $P \in \mathcal{S}$ and $\mathcal{A}, \mathcal{F}$ as before.*

(i)

$$P \textit{ is } \mathcal{F}\textit{-monotone if } Pf \in \mathcal{F} \quad \forall f \in \mathcal{F}$$

(ii) Assuming $\leq_{\mathcal{M}}$ is some stochastic ordering on $\mathcal{M}$.

$$P \textit{ is } (\mathcal{M}, \leq_{\mathcal{M}})\textit{-monotone if } (p \leq_{\mathcal{M}} q \Rightarrow pP \leq_{\mathcal{M}} qP)$$

(iii) Assuming $\leq_S$ is some partial ordering on S.

$$P \textit{ is } (\mathcal{A}, \leq_S)\textit{-monotone if } (s \leq_S t \Rightarrow P1\{A\}(s) \leq P1\{A\}(t) \quad \forall A \in \mathcal{A})$$

It is tempting to find relationships between these definitions. Clearly they depend on the type of stochastic ordering used. Also, we need the following concept. We say that the stochastic ordering $\leq_{\mathcal{M}}$ is closed for $\mathcal{F}$ if for any nonnegative function f holds

$$(p \leq_{\mathcal{F}} q \Rightarrow pf \leq qf) \;\Rightarrow f \in \mathcal{F} \tag{1}$$

Then

Corollary 2 *Assuming $\leq_{\mathcal{F}}$ is an integral stochastic ordering on $\mathcal{M}$.*
(i)

$$P \textit{ is } \mathcal{F}\textit{-monotone } \Rightarrow P \textit{ is } (\mathcal{M}, \leq_{\mathcal{F}})\textit{-monotone}$$

(ii) If $\leq_{\mathcal{F}}$ is closed then

$$P \textit{ is } (\mathcal{M}, \leq_{\mathcal{F}})\textit{-monotone } \Rightarrow P \textit{ is } \mathcal{F}\textit{-monotone}$$

Here is a simple counterexample to show the necessity of the closeness property in (ii).

Example 1

Let $S = \{0,1\}^2$ and $\mathcal{F} = \{f(s) = f_1(s_1)f_2(s_2) : f_k : \{0,1\} \to \mathbb{R}_{\geq 0} \text{ nondecreasing}\}$. Suppose $p \leq_{\mathcal{F}} q$ and choose $f_1(i) = i, f_2(i) = 1$. Then

$$p(10) + p(11) = pf \leq qf = q(10) + q(11)$$

Now choose $f_1(i) = 1, f_2(i) = i$. Then

$$p(01) + p(11) = pf \leq qf = q(01) + q(11)$$

Now choose $g \in \mathcal{V}$ by $g(s) = s_1 + s_2$. Then these two inequalities yield

$$pg = p(10) + p(01) + 2p(11) \leq q(10) + q(01) + 2q(11) = qg$$

However, $g \notin \mathcal{F}$ and hence $\leq_{\mathcal{F}}$ is not closed for $\mathcal{F}$.

Consider the following stochastic matrix

$$P = \begin{array}{c} \\ 00 \\ 10 \\ 01 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 10 & 01 & 11 \end{array} \\ \begin{pmatrix} 1/2 & 1/4 & 1/4 & 0 \\ 1/4 & 3/8 & 1/8 & 1/4 \\ 1/4 & 1/8 & 3/8 & 1/4 \\ 0 & 1/8 & 1/8 & 3/4 \end{pmatrix} \end{array}$$

Now let us show that P is $(\mathcal{M}, \leq_{\mathcal{F}})$-monotone. Suppose $p \leq_{\mathcal{F}} q$ and $f = f_1 f_2 \in \mathcal{F}$. Notice that for $s = (s_1, s_2) \in S$

$$f_k(s_k) = a_k + b_k \mathbb{1}\{1\}(s_k)$$

with $a_k = f_k(0) \geq 0$ and $b_k = f_k(1) - f_k(0) \geq 0$. Hence

$$f(s) = a_1 a_2 + a_2 b_1 \mathbb{1}\{10, 11\}(s) + a_1 b_2 \mathbb{1}\{01, 11\}(s_2) + b_1 b_2 \mathbb{1}\{11\}(s)$$

It is an easy matter of verifying that

$$pP\mathbb{1}\{10, 11\} \leq qP\mathbb{1}\{10, 11\},\ pP\mathbb{1}\{01, 11\} \leq qP\mathbb{1}\{01, 11\},\ pP\mathbb{1}\{11\} \leq qP\mathbb{1}\{11\}$$

Using linearity, we obtain $pPf \leq qPf$, which proves $(\mathcal{M}, \leq_{\mathcal{F}})$-monotonicity.

Finally, consider $f = \mathbb{1}\{11\} \in \mathcal{F}$. Then Pf equals the last column of P which clearly does not lie in $\mathcal{F}$, i.e. P is not $\mathcal{F}$-monotone. $\square$

Notice that the concept of monotonicity as defined in (ii) of definition 2 is more general than as in (i) since it only requires a stochastic ordering on $\mathcal{M}$. In the case of set stochastic ordering which is not supported by a class of functions, no statement about relationships can be made. However, if we generate functions by taking linear combinations of indicator functions, i.e. $\mathcal{F} \subset \Omega(\mathcal{A})$, we can apply corollary 1 to state that the two orderings are equivalent, and corollary 2 to obtain relations of monotonicity.

Now assume that $\leq_v$ is the vector ordering on S and specify the sets of $\mathcal{A}$ as the 'increasing sets':

$$s \in A, s \leq_v t \Rightarrow t \in A$$

The stochastic ordering induced by these sets is denoted by
eq_d. Then it is well known (cf. [2]) that (ii) and (iii) in definition 2 are equivalent:

$$P \text{ is } (\mathcal{M}, \leq_d)\text{-monotone} \ \Leftrightarrow P \text{ is } (\mathcal{A}, \leq_v)\text{-monotone}$$

In this case the class $\mathcal{F}$ consisting of the nonnegative 'nondecreasing functions' on S,

$$s \leq_v t \Rightarrow f(s) \leq f(t)$$

is determined by the increasing sets, $\mathcal{F} = \Omega(\mathcal{A})$ and is closed (cf. [5]). So we obtain the following result.

Corollary 3 *Let $\leq_v$ be the vector ordering on S, $\mathcal{A}$ the collection of increasing sets of S, and $\mathcal{F}$ the class of nonnegative, nondecreasing functions on S.*

$$\begin{aligned} & P \textit{ is } \mathcal{F}\textit{-monotone} \\ \Leftrightarrow\ & P \textit{ is } (\mathcal{M}, \leq_{\mathcal{F}})\textit{-monotone} \\ \Leftrightarrow\ & P \textit{ is } (\mathcal{M}, \leq_{\mathcal{A}})\textit{-monotone} \\ \Leftrightarrow\ & P \textit{ is } (\mathcal{A}, \leq_v)\textit{-monotone} \end{aligned}$$

Generally all these equivalencies are not valid as the following examples show.

Example 2

Let $\leq_v$ be the vector ordering on S. Define the collection $\mathcal{A}$ of threshold sets,

$$A \in \mathcal{A} \Leftrightarrow A = \{s \in S : a \leq_v s\}, a \in S$$

The stochastic ordering induced by these sets will be denoted by $\leq_K$. Next, define the class $\mathcal{G}$ of increasing product functions,

$$g \in \mathcal{G} \Leftrightarrow g = \prod_{m=1}^{M} g_m$$

with $g_m : \{0, 1, \ldots\} \to \mathbb{R}_{\geq 0}$ nondecreasing

Then

$$\begin{aligned} & P \text{ is } \mathcal{F}\text{-monotone} \\ \overset{(a)}{\Rightarrow}\ & P \text{ is } (\mathcal{M}, \leq_{\mathcal{F}})\text{-monotone} \\ \overset{(b)}{\Leftrightarrow}\ & P \text{ is } (\mathcal{M}, \leq_{\mathcal{A}})\text{-monotone} \\ \overset{(c)}{\Rightarrow}\ & P \text{ is } (\mathcal{A}, \leq_v)\text{-monotone} \end{aligned}$$

(a) is by corollary 2

(b) is shown by [1]

(c) is trivial

However, the reverse of (a) is not true as we have shown in example 1. And the reverse of (c) is not true as can be seen in the following counterexample. Let $S = \{0,1\}^2$ and

$$P = \begin{array}{c c} & \begin{array}{cccc} 00 & 10 & 01 & 11 \end{array} \\ \begin{array}{c} 00 \\ 10 \\ 01 \\ 11 \end{array} & \begin{pmatrix} 0.2 & 0.2 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.4 \\ 0.2 & 0.2 & 0.2 & 0.4 \\ 0.2 & 0.2 & 0.2 & 0.4 \end{pmatrix} \end{array}$$

Clearly, $P(s,A) \leq P(t,A)$ whenever $s \leq_v t$ and $A \in \mathcal{A}$, i.e. P is $(\mathcal{A}, \leq_v)$-monotone. Furthermore, define $p, q \in \mathcal{M}$ by

$$p(00) = 0.2,\ p(10) = 0.3,\ p(01) = 0.3,\ p(11) = 0.2$$

$$q(00) = 0.3,\ q(10) = 0.1,\ q(01) = 0.1,\ q(11) = 0.5$$

Then $p(A) \leq q(A)$ for all $A \in \mathcal{A}$, i.e. $p \leq_{\mathcal{A}} q$. However, for $A = \{11\}$,

$$pP(A) = 0.36 > 0.34 = qP(A)$$

i.e. P is not $(\mathcal{M}, \leq_{\mathcal{A}})$-monotone. □

Example 3

As in the previous example, consider S with the $\leq_v$ vector ordering and the collection $\mathcal{A}$ of threshold sets. Now define the so-called Δ-functions,

$$\begin{aligned} \mathcal{F} &= \{f : S \to \mathbb{R}_{\geq 0} : \Delta_{m_1 m_2 \cdots m_r} h \geq 0 \text{ (on } S) \\ &\qquad \forall r = 1, 2, \ldots, M \text{ and } 1 \leq m_1 < m_2 < \cdots < m_r \leq M\} \end{aligned} \tag{2}$$

where for any $r = 1, 2, \ldots, M$ and $1 \leq m_1 < m_2 < \cdots < m_r \leq M$,

$$\Delta_{m_1 m_2 \cdots m_r} : \mathcal{V} \to \mathcal{V}$$

is defined by

$$\begin{aligned}\Delta_{m_1 m_2 \cdots m_r} f(s) &= f(s) - \sum_{i=1}^{r} f(s - e_{m_i}) + \sum_{1 \leq i < j \leq r} f(s - e_{m_i} - e_{m_j}) \\ &- \sum_{1 \leq i < j < k \leq r} f(s - e_{m_i} - e_{m_j} - e_{m_i}) \cdots (-1)^r h(s - e_{m_1} - e_{m_2} - \cdots - e_{m_r})\end{aligned}$$

(set $f(t) = 0$ whenever the argument $t \notin S$). One can find more details on this particular class of functions in [4]. Now,

$$\begin{aligned} & P \text{ is } \mathcal{F}\text{-monotone} \\ \overset{(a)}{\Leftrightarrow} \; & P \text{ is } (\mathcal{M}, \leq_{\mathcal{F}})\text{-monotone} \\ \overset{(b)}{\Leftrightarrow} \; & P \text{ is } (\mathcal{M}, \leq_{\mathcal{A}})\text{-monotone} \\ \overset{(c)}{\Rightarrow} \; & P \text{ is } (\mathcal{A}, \leq_v)\text{-monotone} \end{aligned}$$

(a) and (b) can be found in [4]

(c) is trivial

Again the counterexample of example 2 shows that (c) is not reversed. □

Example 4

Let $S = \{0, 1\}^2$ and define

$$\mathcal{A} = \{\emptyset, S, A_{00} = \{10, 01, 11\}, A_{10} = \{01, 11\}, A_{01} = \{10, 11\}\}$$

and

$$P = \begin{array}{c} \\ 00 \\ 10 \\ 01 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 10 & 01 & 11 \end{array} \\ \begin{pmatrix} 1/2 & 1/6 & 1/6 & 1/6 \\ 1/2 & 1/12 & 0 & 5/12 \\ 1/2 & 0 & 1/6 & 1/3 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix} \end{array}$$

Let us verify that P is $(\mathcal{M}, \leq_{\mathcal{A}})$-monotone by calculating $pP(A)$ for any $p \in \mathcal{M}$ and $A \in \mathcal{A}$.

$$\begin{aligned}
pP(\emptyset) &= 0 \\
pP(S) &= 1 \\
pP(A_{00}) &= 1/2 \\
pP(A_{10} &= 1/3 + (1/12)p(A_{00}) + (1/12)p(A_{10}) \\
pP(A_{01} &= 1/3 + (1/6)p(A_{01})
\end{aligned}$$

This gives readily the $(\mathcal{M}, \leq_{\mathcal{A}})$-monotonicity.

Now define the partial ordering $\leq_S$ on S by

$$00 \leq_S 01 \leq_S 10 \leq_S 11$$

Then

$$P(01, A_{10}) = 1/2 > 5/12 = P(10, A_{10})$$

i.e. P is not $\mathcal{A}, \leq_S)$-monotone. $\square$

4 Comparability of Stochastic Matrices

Comparability of stochastic matrices is defined as follows.

Definition 3 *Let $P, Q \in \mathcal{S}$.*

(i) Assuming $\mathcal{F}$ is a class of functions.

$$P \leq_{\mathcal{F}} Q \Leftrightarrow Pf \leq_{\mathcal{V}} Qf \quad \forall f \in \mathcal{F}$$

(ii) Assuming $\leq_{\mathcal{M}}$ is some stochastic ordering on $\mathcal{M}$.

$$P \leq_{\leq_{\mathcal{M}}} Q \Leftrightarrow pP \leq_{\mathcal{M}} pQ \quad \forall p \in \mathcal{M}$$

In (ii) we explicitely denote the stochastic ordering on $\mathcal{M}$. However, assuming an integral stochastic ordering $\leq_{\mathcal{F}}$ it is easy to see that

$$P \leq_{\leq_{\mathcal{F}}} Q \Leftrightarrow P \leq_{\mathcal{F}} Q \tag{3}$$

Therefore we write simply $P \leq_{\mathcal{F}} Q$ for any of the definitions in case of integral stochastic ordering, and we write $P \leq_{\mathcal{A}} Q$ assuming a set stochastic ordering $\leq_{\mathcal{A}}$ on $\mathcal{M}$. Finally, one can easily check that in the latter case we get

$$P \leq_{\mathcal{A}} Q \Leftrightarrow P1\{A\} \leq_{\mathcal{V}} Q1\{A\} \quad \forall A \in \mathcal{A}$$

5 Stochastic Ordering of Markov Chains

Let $\{X(t) : t = 0, 1, \ldots\}$ and $\{Y(t) : t = 0, 1, \ldots\}$ be two discrete-time Markov chains on S with matrices of transition probabilities $P_X$and P_Y respectively. Set $p_X(t)$ and $p_Y(t)$ for the marginal distributions at time t. For the moment we assume some stochastic ordering $\leq_{\mathcal{M}}$ on the space of probability measures. Commonly one writes

$$X(t) \leq_{\mathcal{M}} Y(t) \tag{4}$$

to mean that the marginal distributions are ordered in the sense of $p_X(t) \leq_{\mathcal{M}} p_Y(t)$. It is well known how to relate (4) to monotonicity and comparability of the transition matrices involved (general reference: [5]).

Lemma 1 *If*

(i)

$$X(t-1) \leq_{\mathcal{M}} Y(t-1)$$

(ii)

$$P_X \text{ or } P_Y \text{ is } \leq_{\mathcal{M}}\text{-monotone}$$

(iii)

$$P_X \leq_{\leq_{\mathcal{M}}} P_Y$$

then (4) holds.

The hard work in problems that rely on application of this result, lies mostly in showing (ii). However, when one assumes the specific stochastic ordering $\leq_d$ induced by the increasing sets, or the stochastic ordering $\leq_K$, induced by the threshold sets, condition (ii) becomes 'fairly simple' because corollaries 1 and 2 are applicable. Corollary 1 says that the set stochastic ordering is equivalent to an integral stochastic ordering. In case of $\leq_d$ the nondecreasing functions (see [5]). In case of $\leq_K$ the Δ-functions (see [4]). With corollary 2 condition (ii) becomes

$$P_X f \in \mathcal{F} \text{ or } P_Y f \in \mathcal{F} \quad \forall f \in \mathcal{F} \tag{5}$$

Let us assume that $\leq_{\mathcal{M}}$ is a set stochastic ordering induced by a collecion of sets $\mathcal{A}$. Inequality (4) expresses stochastic ordering of 1-dimensional marginal distributions of the chains. Generalizing we denote

$$(X(t_1), X(t_2), \ldots, X(t_n)) \leq_{\mathcal{A}} (Y(t_1), Y(t_2), \ldots, Y(t_n)) \tag{6}$$

to mean

$$\begin{aligned} &\mathsf{P}(X(t_1) \in A_1, X(t_2) \in A_2, \ldots, X(t_n) \in A_n) \\ \leq\ &\mathsf{P}(Y(t_1) \in A_1, Y(t_2) \in A_2, \ldots, Y(t_n) \in A_n) \end{aligned}$$

for all $A_1, A_2, \ldots, A_n \in \mathcal{A}$ ($n \in \mathbb{N}$ and $0 \leq t_1 < t_2 < \cdots < t_n$ are arbitrary). Equation (6) expresses stochastic ordering of multi-dimensional marginal distributions of the chains. Generally, the conditions of lemma 1 do not imply this multi-dimensional comparison as the following example shows.

Example 5

Let $S, \mathcal{A}$ and $P_X = P_Y = P$ as in example 4. Define the initial distributions of the chains by

$$p_X(0) = (1/2, 0, 0, 1/2) \quad p_Y(0) = (0, 1/4, 1/2, 1/4)$$

(resp. in the states $00, 10, 01, 11$). Hence, $p_X(0) \leq_{\mathcal{A}} p_Y(0)$. Since P is $(\mathcal{M}, \leq_{\mathcal{A}})$-monotone (see example 4) and $P_X = P_Y$, we get according to lemma 1, $X(t) \leq_{\mathcal{A}} Y(t)$ for any $t = 0, 1, \ldots$. However,

$$\begin{aligned}
\mathsf{P}(X(0) \in A_{01}, X(1) \in A_{10}) &= p_X(0)(11)P(11, \{01, 11\}) = 1/4 \\
&> (1/4)(11/12) = p_Y(0)(10)P(10, \{01, 11\}) + p_Y(0)(11)P(11, \{01, 11\}) \\
&= \mathsf{P}(Y(0) \in A_{01}, Y(1) \in A_{10})
\end{aligned}$$

□

Positive results concerning multi-dimensional comparisons are obtained by specifying the stochastic ordering by increasing or threshold sets.

Theorem 1 *Assume the vector ordering on S. Let the stochastic ordering $\leq_{\mathcal{M}}$ be either induced by the increasing sets ($\leq_d$) or by the threshold sets ($\leq_K$). And*

(i)

$$X(0) \leq_{\mathcal{M}} Y(0)$$

(ii)

P_X *or* P_Y *is* $(\mathcal{M}, \leq_{\mathcal{M}})$*-monotone*

(iii)

$$P_X \leq_{\leq_{\mathcal{M}}} P_Y$$

then (6) holds.

Proof

See [5] for the $\leq_d$ case and [4] for the $\leq_K$ case. □

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